

Return, Risk and Risk Aversion

Topics 1&2
Chapters 5&6, Bodie, Kane&Marcus
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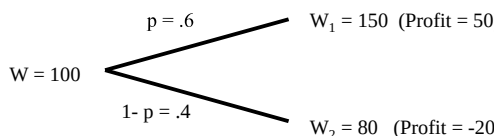
Main Points

- Two broad tasks in portfolio formation
 - Security and market analysis
 - *Formation of an optimal portfolio of assets*
- Three themes in portfolio theory, centering on risk
 - Risk premium, return and risk
 - Portfolio vs. individual asset
 - Risk aversion and utility function

Risk with Simple Prospects

- Risk means more than one outcome is possible
- A *simple prospect* is an investment opportunity with a certain initial wealth and two possible outcomes

Risk with Simple Prospects

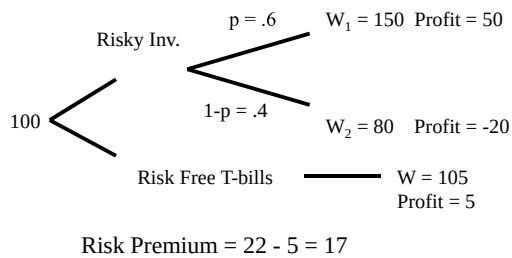


$E(W) = pW_1 + (1-p)W_2 = .6(150) + .4(80) = 122$
 (expected profit = 22)

$\sigma^2 = p[W_1 - E(W)]^2 + (1-p)[W_2 - E(W)]^2$
 $= .6(150-122)^2 + .4(80-122)^2 = 1,176$

$\sigma = 34.293$

Risky Investments with Risk-Free



Holding Period Returns

Rates of Return: Single Period

$$HPR = \frac{P_1 - P_0 + D_1}{P_0}$$

HPR = Holding Period Return

P_0 = Beginning price

P_1 = Ending price

D_1 = Dividend during period one

Rates of Return: Single Period Example

Ending Price = 48
 Beginning Price = 40
 Dividend = 2

$$HPR = (48 - 40 + 2) / (40) = 25\%$$

Expected Returns for A Single Asset

Expected returns

$$E(r) = \sum_s p(s)r(s)$$

$p(s)$ = probability of a state

$r(s)$ = return if a state occurs (For historical information, you can use HPR here.)

s = state

Scenario Returns: Example

<u>State</u>	<u>Prob. of State</u>	<u>r in State</u>
1	.1	-.05
2	.2	.05
3	.4	.15
4	.2	.25
5	.1	.35

$$E(r) = (.1)(-.05) + (.2)(.05) + (.4)(.15) + (.2)(.25) + (.1)(.35)$$

$$E(r) = .15$$

Variance for A Single Asset

Variance:

$$\sigma^2 = \sum_s p(s) [r(s) - E(r)]^2$$

$$\text{Standard deviation} = [\text{variance}]^{1/2}$$

Using Our Example:

$$\text{Var} = [(.1)(-.05-.15)^2 + (.2)(.05-.15)^2 + (.4)(.15-.15)^2 + (.2)(.25-.15)^2 + (.1)(.35-.15)^2]$$

$$\text{Var} = .012$$

$$\text{S.D.} = [.012]^{1/2} = .1095$$

Time Series Analysis of Past Rates of Return

Expected Returns and the Arithmetic Average
(assuming the equal probability)

$$E(r) = \sum_{s=1}^n p(s)r(s) = \frac{1}{n} \sum_{s=1}^n r(s)$$

Geometric (Time-weighted) Average Return

$$TV = (1+r_1)(1+r_2)\dots(1+r_n) = (1+g)^n$$

TV = Terminal Value of the Investment

$$g = TV^{1/n} - 1$$

g = geometric average rate of return

Averaging Returns

- Geometric mean considers the principle of compounding, and we assume that dividend are reinvested
- Arithmetic mean will always be greater than or equal to the geometric mean

$$r_G \approx r_A - \frac{1}{2} \sigma^2$$

Variance and Standard Deviation

- Variance = expected value of squared deviations (assuming equal probability)

$$\sigma^2 = \frac{1}{n} \sum_{s=1}^n [r(s) - \bar{r}]^2$$

- When we deal with a sample instead of the population, in order to eliminate the bias, we use (n-1) instead of n in the above formula
- Standard deviation is the square root of variance

Real Life Example

- How to calculate the return and standard deviation (S.D.) for your stock?
- Price — returns — mean of returns — S.D.
- $r = (P_{t+1} - P_t) / P_t$ (HPR) or $r = \ln (P_{t+1} / P_t)$
- Arithmetic Mean Return

$$\bar{r} = \frac{1}{n} \sum_{i=1}^n r_i$$

- S.D.

$$\text{Variance} = \sum (r - \bar{r})^2 / (n-1)$$

$$\text{S.D.} = \sqrt{\text{Variance}}$$

The Reward-to-Volatility (Sharpe) Ratio

$$\text{Sharpe Ratio for Portfolios} = \frac{\text{Risk Premium}}{\text{SD of Excess Return}}$$

Figure 5.4 The Normal Distribution

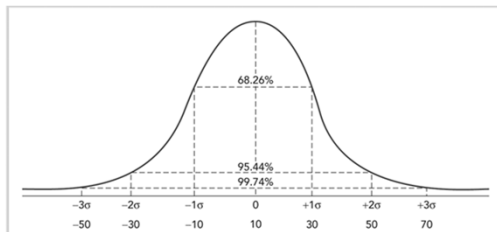


FIGURE 5.4 The normal distribution

Figure 5.5A Normal and Skewed Distributions (mean = 6%, SD = 17%)

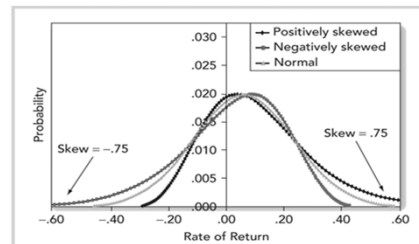


FIGURE 5.5A Normal and skewed distributions (mean = 6%, SD = 17%)

Figure 5.5B Normal and Fat-Tailed Distributions (mean = .1, SD = .2)

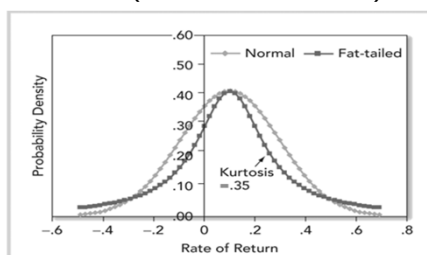


FIGURE 5.5B Normal and fat-tailed distributions (mean = .1, SD = .2)

Asset vs. Portfolio Risk

- When evaluating the risk of a portfolio, we need to take account of the interplay among asset returns
- Two ways to control risk
 - Hedging—investing in an asset with a payoff pattern that offsets exposure to a particular source of risk (insurance contracts)
 - Diversification—putting eggs into many baskets

A Review of Portfolio Math

Example:

Humanex has 50% invested in Best Candy stock, whose value is sensitive to the price of sugar, and another 50% invested in T-bills, which has a return of 5%. When price of sugar rises, Best Candy suffers considerable losses. We have the following scenarios:

	Bull Market	Bear Market	Sugar Crisis
Probability	.5	.3	.2
Return	25%	10%	-25%

A Review of Portfolio Math

Rule 1 : The mean or expected return for an asset is the probability weighted average of its return in all scenarios.

$$E(r) = \sum_s \text{Pr}(s)r(s)$$

$$E(r) = (.5 * 25) + (.3 * 10) + .2(-25) = 10.5\%$$

A Review of Portfolio Math

Rule 2: The variance of an asset's return is the expected value of the squared deviations from the expected return.

$$\sigma^2 = \sum_s \text{Pr}(s)[r(s) - E(r)]^2$$

$$\sigma^2 = .5(25 - 10.5)^2 + .3(10 - 10.5)^2 + .2(-25 - 10.5)^2$$

$$= 357.25$$

$$\sigma = 18.9\%$$

A Review of Portfolio Math

■ When you assume the probability is equal, rules 1&2 becomes:

□ Arithmetic Mean Return

$$\bar{r} = \frac{1}{n} \sum_{i=1}^n r_i$$

□ S.D.

$$\text{Variance} = \sum (r - \bar{r})^2 / (n-1)$$

$$S.D. = \sqrt{\text{Variance}}$$

A Review of Portfolio Math

Rule 3: The rate of return on a portfolio is a weighted average of the rates of return of each asset comprising the portfolio, with the portfolio proportions as weights.

$$r_p = W_1 r_1 + W_2 r_2$$

W_1 = Proportion of funds in Security 1

W_2 = Proportion of funds in Security 2

r_1 = Expected return on Security 1

r_2 = Expected return on Security 2

$$r_p = (.5 \times 10.5) + (.5 \times 5) = 7.75\%$$

A Review of Portfolio Math

Rule 4: When a risky asset is combined with a risk-free asset, the portfolio standard deviation equals the risky asset's standard deviation multiplied by the portfolio proportion invested in the risky asset.

$$\sigma_p = w_{\text{riskyasset}} \times \sigma_{\text{riskyasset}}$$

$$\sigma_p = .5 \times 18.9 = 9.45\%$$

A Review of Portfolio Math

- By reducing the exposure to Best by half and investing in T-bill by half, Humanex reduces both its portfolio return and risk
 - Return: from 10.5% to 7.75%
 - Risk premium: from 5.5% to 2.75% (one half)
 - Standard deviation: from 18.9% to 9.45% (one half)

A Review of Portfolio Math

Example—continued

Another company, Sugarcane, has very different return scenarios from Best, with expected return of 6% and S.D. of 14.73%. Different return pattern offers good diversification potential.

	Bull Market	Bear Market	Sugar Crisis
Probability	.5	.3	.2
Return	1%	-5%	35%

A Review of Portfolio Math

If we invest 50% in Best and 50% in Sugarcane, the portfolio will have an expected return of 8.25%, and a standard deviation of 4.83%.

	Bull Market	Bear Market	Sugar Crisis
Probability	.5	.3	.2
Best	25%	10%	-25%
Sugarcane	1%	-5%	35%
Portfolio	13%	2.5%	5%

$$\text{Portfolio return} = (.5 \times 13) + (.3 \times 2.5) + (.2 \times 5) = 8.25\%$$

A Review of Portfolio Math

Rule 5: When two risky assets with variances σ_1^2 and σ_2^2 , respectively, are combined into a portfolio with portfolio weights w_1 and w_2 , respectively, the portfolio variance is given by:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \text{Cov}(r_1 r_2)$$

$\text{Cov}(r_1 r_2)$ = Covariance of returns for Security 1 and Security 2

A Review of Portfolio Math

- Covariance measures how much the returns on two risky assets move in tandem

□ *Positive* covariance means returns tend to move *together*

□ *Negative* covariance means returns tend to move *inversely*

$$\text{Cov}(A, B) = \sum_s \text{Pr}(s) [r_A(s) - E(r_A)] [r_B(s) - E(r_B)]$$

A Review of Portfolio Math

With $E(r_{\text{Best}}) = 10.5\%$, and $E(r_{\text{Cane}}) = 6\%$

	Bull Market	Bear Market	Sugar Crisis
Probability	.5	.3	.2
Best	25%	10%	-25%
Sugarcane	1%	-5%	35%
Portfolio	13%	2.5%	5%

$$\text{Cov}(r_{\text{Best}}, r_{\text{Cane}}) = .5(25 - 10.5)(1 - 6) + .3(10 - 10.5)(-5 - 6) + .2(-25 - 10.5)(35 - 6) = -240.5$$

A Review of Portfolio Math

- Correlation coefficient is an easier statistics to interpret than the covariance
- Correlation coefficient changes from -1 to 1

$$\rho(\text{Best}, \text{Sugarcane}) = \frac{\text{Cov}[r_{\text{best}}, r_{\text{sugarcane}}]}{\sigma_{\text{Best}} \sigma_{\text{Sugarcane}}} = \frac{-240.5}{18.9 * 14.73} = -.86$$

- This large negative correlation (close to -1) confirms the strong tendency of Best and Sugarcane stocks to move inversely

A Review of Portfolio Math

Rule 5: When two risky assets with variances σ_1^2 and σ_2^2 , respectively, are combined into a portfolio with portfolio weights w_1 and w_2 , respectively, the portfolio variance is given by:

$$\begin{aligned}\sigma_p^2 &= w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \text{Cov}(r_1, r_2) \\ &= .5^2 * 18.9 * 18.9 + .5^2 * 14.73 * 14.73 + \\ &\quad 2 * .5 * .5 * (-240.5) = 23.3 \\ \sigma_p &= 4.83\%\end{aligned}$$

A Review of Portfolio Math

- Rule 5 highlights the effect of covariance on portfolio risk
 - *More positive covariance increases portfolio variance*
 - *More negative covariance decreases portfolio variance*

A Review of Portfolio Math

Portfolio	Expected Return	S.D.
All in Best Candy	10.5%	18.9%
Half in T-bill	7.75	9.45
Half in Sugarcane	8.25	4.83

- With negative correlation, portfolio with Sugarcane dominates the portfolio with T-bills
- The risk of individual assets in a portfolio must be measured in the context of the effect of their return on overall portfolio

Risk and Risk Aversion

- Speculation
 - Considerable risk
 - Sufficient to affect the decision
 - Commensurate gain
- Gamble
 - Bet or wager on an uncertain outcome

Risk Aversion & Utility

- Fair game—a prospect with 0 risk premium
- Investor's view of risk
 - **Risk averse**—only consider risk free or speculative prospects with positive risk premium
 - Risk neutral—don't care about risk
 - Risk seeking—love risk

Risk Aversion & Utility

- Utility—score to rank investment portfolios based on the expected return and risk of portfolios
- Higher return, higher utility; higher volatility, lower utility
- Utility Function

$$U = E(r) - .005 A \sigma^2$$

(use 0.005 when the return and risk are in the percentage form, otherwise use 0.5 as in the book)
- 'A' measures the degree of risk aversion, more risk aversion, larger 'A' will be (what affect 'A'?)

Utility Function

$$U = E(r) - \frac{1}{2} A \sigma^2$$

Where

U = utility

$E(r)$ = expected return on the asset or portfolio

A = coefficient of risk aversion

σ^2 = variance of returns

Risk Aversion & Utility

$$U = E(r) - .005 A \sigma^2$$

$$= 22\% - .005 A (34\%)^2$$

Risk Aversion A Value

High 5 -6.90

↓ 3 4.66

Low 1 16.22

} T-bill = 5%

You will only invest when the value is higher than risk free rate.

Risk Aversion & Utility

- *Risk averse* investors only accept investments with *positive risk premium* (**considering both return and risk; $A > 0$**)
- *Risk neutral* investors judge risky prospects solely by their *expected rate of return* (**considering return only; $A = 0$**)
- *Risk lovers* adjust the expected return upward to take into account the 'fun' of taking risk (**risk becomes positive effect; $A < 0$**)

Available Risky Portfolios ($R_f = 5\%$)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

TABLE 6.1

Available risky portfolios (Risk-free rate = 5%)

Utility Scores of Alter. Portfolios for Investors with Varying Risk Aversion

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r) = .07; \sigma = .05$]	Utility Score of Portfolio M [$E(r) = .09; \sigma = .10$]	Utility Score of Portfolio H [$E(r) = .13; \sigma = .20$]
2.0	$.07 - \frac{1}{2} \times 2 \times .05^2 = .0675$	$.09 - \frac{1}{2} \times 2 \times .1^2 = .0800$	$.13 - \frac{1}{2} \times 2 \times .2^2 = .09$
3.5	$.07 - \frac{1}{2} \times 3.5 \times .05^2 = .0656$	$.09 - \frac{1}{2} \times 3.5 \times .1^2 = .0725$	$.13 - \frac{1}{2} \times 3.5 \times .2^2 = .06$
5.0	$.07 - \frac{1}{2} \times 5 \times .05^2 = .0638$	$.09 - \frac{1}{2} \times 5 \times .1^2 = .0650$	$.13 - \frac{1}{2} \times 5 \times .2^2 = .03$

TABLE 6.2

Utility scores of alternative portfolios for investors with varying degrees of risk aversion

Mean-variance Criterion

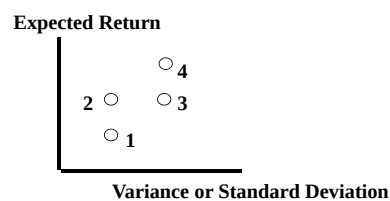
- Risk averse investors like investment with high return and low risk
- A dominates B if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

Dominance Principle



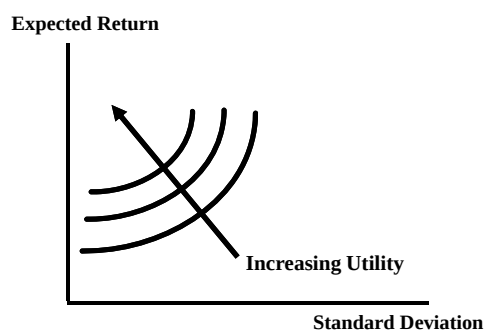
- 2 dominates 1; 2 has a higher return
- 2 dominates 3; 2 has a lower risk
- 4 dominates 3; 4 has a higher return
- How about choice between 1 and 3?

Utility and Indifference Curves

- Indifference curves represent an investor's willingness to trade-off return and risk
- Investments on one indifference curve will have high return, high risk; or low return, low risk, but with the same utility score
- Example (A=4)

<u>Exp Ret</u>	<u>St Deviation</u>	<u>$U = E(r) - .005A\sigma^2$</u>
10%	20.0%	2
15	25.5	2
20	30.0	2
25	33.9	2

Indifference Curves





Example--1

- Stock X has a standard deviation of return of 10%. Stock Y has a standard deviation of return of 20%. The correlation coefficient between stocks is 0.5. If you invest 60% of the funds in stock X and 40% in stock Y, what is the standard deviation of a portfolio?
 - A) 10%
 - B) 20%
 - C) 12.2%
 - D) 22%
 - E) None of the above



Example--1

- Answer: C
- Response: $(0.6^2)(10^2) + (0.4^2)(20^2) + (2)(0.6)(0.4)(0.5)(10)(20) = 148$;
Standard deviation = $(148^{0.5}) = 12.2\%$



Example--2

- Use the information for question 1-2:
Florida Company (FC) and Minnesota Company (MC) are both service companies. Their historical returns for the past three years are: FC: -10%, 15%, 25%; MC: 10%, 6%, 32%.
- 1. If FC and MC are combined in a portfolio with 50% of the funds invested in each, calculate the expected return on the portfolio.
 - A) 10%
 - B) 13%
 - C) 16%
 - D) None of the above



Example 2

- Florida Company (FC) and Minnesota Company (MC) are both service companies. Their historical returns for the past three years are: FC: -10%, 15%, 25%; MC: 10%, 6%, 32%.
- 2. What is the variance of the portfolio with 50% of the funds invested in FC and 50% in MC (approximately)?
 - A) 208
 - B) 325
 - C) 188
 - D) None of the above



Example--2

- Answer 1: B

$$ER(FC) = (-10 + 15 + 25) / 3 = 10\%$$

$$ER(MC) = (10 + 6 + 32) / 3 = 16\%$$

$$ER(\text{Portfolio}) = 0.5 * 10 + 0.5 * 16 = 13\%$$

- Answer 2: A

$$\text{Cov} = [(-10-10)(10-16) + (15-10)(6-16) + (25-10)(32-16)] / (3-1) = (120-50+240) / (3-1) = 155$$

$$\text{Var}(FC) = [(-10-10)^2 + (15-10)^2 + (25-10)^2] / (3-1) = 325$$

$$\text{Var}(MC) = [(10-16)^2 + (6-16)^2 + (32-16)^2] / (3-1) = 196$$

$$\text{Var}(\text{Portfolio}) = (0.5^2)(325) + (0.5^2)(196) + (2)(0.5)(0.5)(155) = 208$$